Supersymmetry and the gauge/gravity duality

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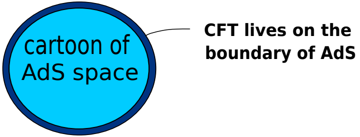
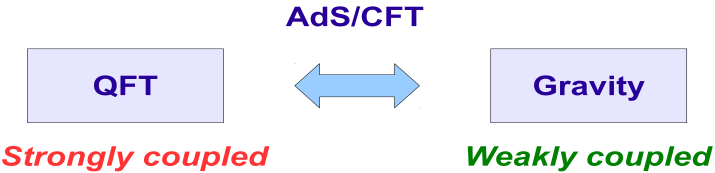
Outline

Gauge/Gravity duality

Formerly known as AdS/CFT correspondence

Conjectured equivalence between (quantum) **gravity** in certain "bulk" space-times and **quantum field theories** on their boundaries





Supersymmetry

- When bulk and boundary are supersymmetric we can perform detailed computations on both sides and (in certain limits) compare them
- Supersymmetry in the bulk $\Rightarrow$ **supersymmetric solutions of supergravity equations**
- There exist spinor fields (Killing spinors) obeying linear first order differential equations (KSE)
- Supersymmetry on the boundary $\Rightarrow$ **"rigid" KSE on curved space***

*[See talk of Zaffaroni]

Field theories from M2-branes

Over the last four years lot of progress in the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence

- Constructions of large classes of $\mathbf{d = 3}$ superconformal field theories with known gravity duals
- Precise quantitative checks of the gauge/gravity duality

The $\mathbf{d = 3}$ supersymmetric field theories describe the dynamics of $\mathbf{N}$ M2-branes, with supergravity dual description valid in the large $\mathbf{N}$ limit

Field theories from M2-branes

[ABJM]: worldvolume theory on **N** M2-branes in flat spacetime

The key was to study **N** M2-branes on $\mathbb{R}^{1,2} \times \mathbb{R}^8/\mathbb{Z}_k$, where the $\mathbb{Z}_k$ quotient leaves $\mathcal{N} = 6 \subset \mathcal{N} = 8$ supersymmetry unbroken

Low-energy theory is an $\mathcal{N} = 6$ superconformal $U(N)_k \times U(N)_{-k}$ Chern-Simons theory coupled to bifundamental matter, with $k \in \mathbb{N}$ a Chern-Simons coupling:

$$S_{CS} = \frac{k}{4\pi} \int \mathrm{Tr}\left(\mathcal{A} \wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right) + \text{supersymmetry completion}$$

M-theory dual of ABJM model

Gravity dual:

$\mathrm{AdS}_4 \times \mathbf{S}^7/\mathbb{Z}_k$ solution to $\mathbf{d = 11}$ supergravity with quantized flux of $\mathbf{G}$:

$$\mathbf{N} = \frac{1}{(2\pi\ell_p)^6}\int_{S^7/\mathbb{Z}_k} *\mathbf{G}$$

$\mathbf{3/4}$ unbroken supersymmetry

$\mathbf{N}$ is the number of M2 branes $=$ $\mathbf{N}$ in $\mathbf{U(N)}$

$\mathbf{k}$ is the Chern-Simons level

Generalisations with less supersymmetry

- M2-branes at other isolated singularities in 8 dimensions: $\mathbb{R}^{1,2} \times \mathbf{X_8}$ with $\mathbf{X_8}$ hyper-Kähler ($\mathcal{N} = \mathbf{3}$) or Calabi-Yau ($\mathcal{N} = \mathbf{2}$)
- Field theories: Chern-Simons-matter theories with products of $\mathbf{U(N)}$ gauge groups ("quivers")
- Conical metric $\mathbf{ds^2_{X_8} = dr^2 + r^2 ds^2_{Y_7}}$ leads to supergravity dual $\mathrm{AdS}_4 \times \mathbf{Y_7}$, with $\mathbf{Y_7}$ a Sasaki-Einstein (or three-Sasakian) manifold

The boundary of Euclidean AdS$_4$

The conformal boundary of Euclidean-AdS$_4$ is $\mathbf{S^3}$ with "round" (Einstein) metric

One can put an arbitrary $\mathbf{d = 3}$, $\mathcal{N} = \mathbf{2}$ gauge theory on the round $\mathbf{S^3}$, preserving supersymmetry [Kapustin-Willet-Yaakov, Jafferis, Hama-Hosomichi-Lee]

Key: on the round $\mathbf{S^3}$ there exist Killing spinors

$$\text{Flat space} \quad \partial_\mu \epsilon = 0 \quad \longrightarrow \quad \text{curved space} \quad \nabla_\mu \epsilon = \frac{i}{2} \gamma_\mu \epsilon$$

[Festuccia-Seiberg]: begin with supergravity, take $\mathbf{m_{pl}} \to \infty$ limit to obtain a rigid supersymmetric theory

Localization

[See also talk of Belavin]

In Euclidean supersymmetric theories on $\mathbf{S}^3$ the VEV of any BPS operator can be computed exactly using localization [Pestun]

Basic idea: if there is a Fermionic symmetry $\mathcal{Q}$ ($\mathcal{Q}^2 = 0$) such that $\mathcal{Q}\mathbf{S} = \mathcal{Q}\mathcal{O}_{\mathbf{BPS}} = 0$, then the semi-classical limit becomes exact

$$\langle \mathcal{O}_{\mathbf{BPS}} \rangle = \int_{\text{all fields}} e^{-\mathbf{S}} \mathcal{O}_{\mathbf{BPS}}$$

$$\overset{\text{exactly}}{=} \int_{\mathcal{Q}-\text{invariant fields}} e^{-\mathbf{S}} \mathcal{O}_{\mathbf{BPS}} \cdot (\text{one-loop determinant})$$

On supersymmetric (admitting Killings spinors) curved Euclidean manifolds $\mathcal{Q}$ is a supercharge, generating a supersymmetry variation of the theory

For $\mathbf{d} = \mathbf{3}$, $\mathcal{N} = \mathbf{2}$ $\mathbf{U(N)}$ gauge theory, infinite-dimensional functional integral $\longrightarrow$ finite-dimensional matrix integral over $\mathbf{N} \times \mathbf{N}$ Hermitian matrices

The localized partition function

(Here written for $\mathbf{U(N)_k}$ gauge group with one fundamental matter for simplicity)

$$Z \propto \int \prod_{j=1}^{N} \frac{d\lambda_j}{2\pi} \exp\left[i \frac{k}{4\pi} \sum_{j=1}^{N} \lambda_j^2 \right] \exp\left[-F_{\text{loop}}\right]$$

where

$$\exp\left[-F_{\text{loop}}\right] = \prod_{m \neq j} 2 \sinh\left(\frac{\lambda_m - \lambda_j}{2}\right) \cdot \prod_{j=1}^{N} s_{b=1}(i - i\Delta - \lambda_j)$$

$s_b(x)$ is the double sine/quantum dylogarithm function

$$s_b(x) = \prod_{m,n \geq 0} \frac{mb + nb^{-1} + (b + b^{-1})/2 - ix}{mb + nb^{-1} + (b + b^{-1})/2 + ix}$$

[See the talks of Faddeev and Spiridonov!]

Exact free energy

The partition function **Z** matrix integral in simple cases may be computed explicitly. For the ABJM model [Drukker-Marino-Putrov]:

$$-\log \mathbf{Z}_{\text{field theory}} = \frac{\pi\sqrt{2}}{3}\mathbf{k}^{1/2}\mathbf{N}^{3/2} + \mathbf{O}(\mathbf{N}^{1/2})$$

This agrees precisely (i.e. including numerical factors!) with the holographic free energy of AdS_4 (holographically renormalized action of AdS_4), reproducing the famous $\mathbf{N}^{3/2}$ scaling

Analytic and/or numerical methods may be used to compute $\mathbf{1/N}$ corrections

Large $\mathbf{N}$ free energy

For more general $\mathcal{N} = 2$ SCFTs, similar results have been obtained by computing the large $\mathbf{N}$ limit of matrix integrals:

$$-\log \mathbf{Z}_{\text{field theory}} = \sqrt{\frac{2\pi^6}{27\mathrm{Vol}(\mathbf{Y}_7)}}\mathbf{N}^{3/2} + \mathbf{O}(\mathbf{N}^{1/2})$$

(at least when the matter representation of the gauge group is *real*)

This agrees with the holographic free energy computed from the (Euclidean) M-theory solutions $\mathrm{AdS}_4 \times \mathbf{Y}_7$!

[DM-Sparks, Jafferis-Klebanov-Pufu-Safdi, Cheon-Kim-Kim]

More general three-manifolds

One can put $\mathcal{N} = 2$ SUSY theories on 3-manifolds more general than the round S^3, still preserving supersymmetry ([Klare-Tomasiello-Zaffaroni]). General rigid KSE given in [Closset-Dumitrescu-Festuccia-Komardgodski]

$$\left[\nabla_\alpha - iA^{(3)}_\alpha + iV_\alpha + \frac{H}{2}\gamma_\alpha + \frac{1}{2}\epsilon_{\alpha\beta\rho}V^\beta\gamma^\rho\right]\chi = 0$$

$A^{(3)}_\alpha, V_\alpha, H$ are fixed (rigid) background fields

Our main example: results about supersymmetry, localization, and reduction to matrix integrals go through if we replace the round S^3 by the biaxially squashed S^3, with metric

$$ds^2_3 \;=\; d\theta^2 + \sin^2\theta d\phi^2 + 4s^2\left(d\psi + \cos\theta d\phi\right)^2$$

and a background $U(1)_R$ gauge fields $A^{(3)}$

$$\text{Flat space} \quad \partial_\mu - iq\mathcal{A}_\mu \quad \longrightarrow \quad \text{curved space} \quad \nabla_\mu - iq\mathcal{A}_\mu - irA^{(3)}{}_\mu$$

The two SUSY biaxially squashed three-spheres

Supersymmetry can be preserved in two cases, adding slightly different background gauge fields:

$$\begin{aligned} 1/4 \text{ BPS:} \quad \mathbf{A}^{(3)} &= -\frac{1}{2}(4s^2-1)\,(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi) \qquad \text{[Hama-Hosomichi-Lee]} \\ 1/2 \text{ BPS:} \quad \mathbf{A}^{(3)} &= -s\sqrt{4s^2-1}\,(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi) \qquad \text{[Imamura-Yokoyama]} \end{aligned}$$

Here $0 < s =$ *squashing parameter*, with the round metric on S^3 being $s = \frac{1}{2}$

In the 1/2 BPS case the partition function involves $s_b(x)$, where $b = b(s)$

The large N limit of the partition function for $d = 3$, $\mathcal{N} = 2$ theories can be computed from the matrix models and to leading order in N is:

$$\log Z_{\text{field theory}}[s] = \log Z_{\text{round } S^3} \times \begin{cases} 1 & 1/4 \text{ BPS} \\ 4s^2 & 1/2 \text{ BPS} \end{cases}$$

Gravity duals

Idea: find a supersymmetric filling $\mathbf{M_4}$ of the squashed $\mathbf{S^3}$ in $\mathbf{d=4}$, $\mathcal{N}=2$ gauged supergravity (Einstein-Maxwell theory), and use the fact that any[1] such solution uplifts to a supersymmetric solution $\mathbf{M_4 \times Y_7}$ of $\mathbf{d=11}$ supergravity

Action: $\mathbf{S} = -\frac{1}{16\pi G_4}\int d^4x\sqrt{g}\left(R+6-F^2\right)$

Killing spinor equation: $\left(\nabla_\mu - iA_\mu + \frac{1}{2}\Gamma_\mu + \frac{i}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu\right)\epsilon = 0$

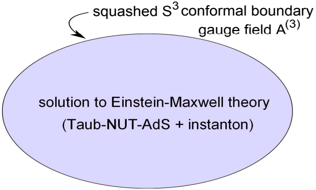
Where $\Gamma_\mu \in \mathrm{Cliff}(4,0)$, so $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$

Dirichlet problem: find an $(\mathbf{M_4}, g_{\mu\nu})$ and gauge field $\mathbf{A}$ such that

- The conformal boundary of $\mathbf{M_4}$ is the squashed sphere
- The $\mathbf{d=4}$ gauge field $\mathbf{A}$ restricts to $\mathbf{A^{(3)}}$ on the conformal boundary
- The $\mathbf{d=4}$ Killing spinor ϵ restricts to the $\mathbf{d=3}$ Killing spinor χ

[1] We shall see...

Gravity duals



$\mathbf{M_4}$ = Taub-NUT-AdS

A = self-dual gauge field (*F=F)

The gauge fields and Killing spinors are different for the 1/4 BPS and 1/2 BPS solutions

Taub-NUT-AdS is an asymptotically locally AdS Einstein metric (with self-dual Weyl tensor) on $\mathbb{R}^4$:

$$\mathrm{d}s_4^2 = \frac{r^2 - s^2}{\Omega(r)}\mathrm{d}r^2 + (r^2 - s^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) + \frac{4s^2\Omega(r)}{(r^2 - s^2)}(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi)^2$$

where $\Omega(r) = (r - s)^2\,[1 + (r - s)(r + 3s)]$

$$A = f(r, s)(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi)$$

Holographic free energy

The holographic free energy is

$$-\log Z_{\text{gravity}} \;=\; S_{\text{Einstein-Maxwell}} + S_{\text{Gibbons-Hawking}} + S_{\text{counterterm}}$$

Remarkably, we find

$$\log Z_{\text{gravity}}[s] \;=\; \log Z_{\text{AdS}_4} \times \begin{cases} 1 & 1/4 \text{ BPS} \\ 4s^2 & 1/2 \text{ BPS} \end{cases}$$

agreeing exactly with the leading large **N** matrix model results!

For the 1/4 BPS case the independence of **s** is non-trivial: each term in the action has a complicated **s**-dependence, which cancels only when all are summed

Is the filling of a boundary unique?

We must *sum* over all solutions $\mathbf{M_4^{(i)}}$ with fixed conformal boundary $\partial\mathbf{M_4^{(i)}} = \mathbf{M_3}$:

$$\mathbf{Z_{\text{gravity}} = \sum_i \exp[-S(M_4^{(i)})]}$$

$\mathbf{S(M_4^{(i)}) \sim N^{3/2}} \Rightarrow$ the large $\mathbf{N}$ limit, the solution with smallest free energy/Euclidean action dominates (exponentially) the saddle point

We can use the fact that the biaxially squashed $\mathbf{S^3}$ metric and background gauge field $\mathbf{A^{(3)}}$ preserve $\mathbf{SU(2) \times U(1)}$ symmetry

A theorem of [Anderson] guarantees this symmetry extends to a symmetry of $\mathbf{M_4}$ if the four dimensional metric is Einstein, and more generally conjectures it for solutions to Einstein-Maxwell

Assuming this, we have *completely solved* this filling problem

NUTs and bolts

More generally, we have studied biaxially squashed Lens space $\mathbf{S^3}/\mathbb{Z}_\mathbf{p}$ as conformal boundary, which still preserve $\mathbf{SU(2)} \times \mathbf{U(1)}$

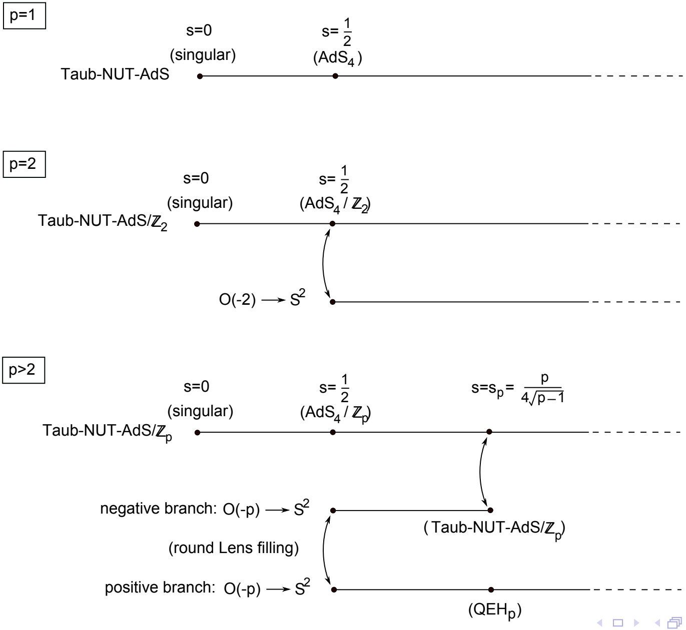
We have written down all AlEAdS supersymmetric solutions of Maxwell-Einstein supergravity. The results are surprisingly complicated!

- The Taub-NUT-AdS solutions are the unique supersymmetric solutions with topology $\mathbb{R}^\mathbf{4}$, and extend to (mildly singular) $\mathbb{R}^\mathbf{4}/\mathbb{Z}_\mathbf{p}$ solutions by quotienting
- There is another supersymmetric Einstein (Weyl self-dual) solution, with 1/4 BPS and 1/2 BPS instantons, of topology $\boldsymbol{\mathcal{M}_\mathbf{p} = \mathcal{O}(-\mathbf{p}) \to \mathbf{S^2}}$, which exists for $\mathbf{p} \geq \mathbf{3}$ and $\mathbf{s} = \mathbf{s_p} \equiv \frac{\mathbf{p}}{\mathbf{4}\sqrt{\mathbf{p}-\mathbf{1}}}$ ("Quaternionic-Eguchi-Hanson")
- In fact, this is a special case, for fixed $\mathbf{s} = \mathbf{s_p}$, of a more general class of supersymmetric Taub-Bolt-AdS solutions, of topology $\boldsymbol{\mathcal{M}_\mathbf{p}}$ for $\mathbf{p} \geq \mathbf{1}$

Supersymmetric Taub-Bolt-AdS solutions

- Exist for both 1/2 BPS and 1/4 BPS boundary conditions, but only for certain *ranges* of squashing $\mathbf{s} \in [\mathbf{s}_-, \mathbf{s}_+]$
- In general there are different branches of solutions, joining onto each other at special solutions (such as the endpoints $\mathbf{s}_\pm$)
- The manifolds $\mathcal{M}_\mathbf{p} = \mathcal{O}(-\mathbf{p}) \to \mathbf{S}^2$ are not spin manifolds for $\mathbf{p}$ odd, but the gauged supergravity spinors are global, smooth spin$^\mathbf{c}$ spinors
- Correspondingly, we find that the metric being smooth implies that the gauge field $\mathbf{A}$ is automatically a *quantized spin*$^\mathbf{c}$ *connection*:

$$\int_{S^2} \frac{\mathbf{F}}{2\pi} = \begin{cases} \pm\frac{\mathbf{p}}{2} & 1/2 \text{ BPS} \\ \pm\frac{\mathbf{p}}{2} - 1 & 1/4 \text{ BPS} \end{cases}$$


p=1
s=0
(singular)
Taub-NUT-AdS
$s=\frac{1}{2}$
(AdS_4)
p=2
s=0
(singular)
Taub-NUT-AdS/$\mathbb{Z}_2$
$s=\frac{1}{2}$
($AdS_4 / \mathbb{Z}_2$)
O(-2) ⟶ S^2
p>2
s=0
(singular)
Taub-NUT-AdS/$\mathbb{Z}_p$
$s=\frac{1}{2}$
($AdS_4 / \mathbb{Z}_p$)
$s=s_p=\frac{p}{4\sqrt{p-1}}$
negative branch: O(-p) ⟶ S^2
(Taub-NUT-AdS/$\mathbb{Z}_p$)
(round Lens filling)
positive branch: O(-p) ⟶ S^2
(QEH_p)

1/2 BPS

Holographic free energies of 1/2 BPS solutions

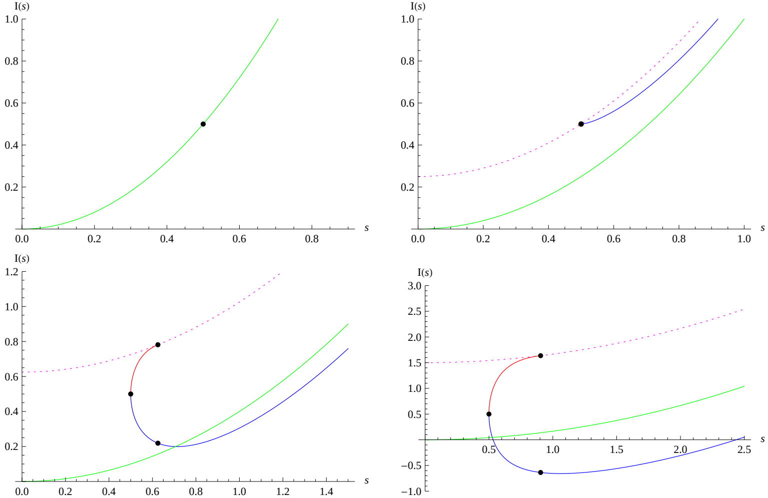


Figure: Plots of the free energies **I(s)** as functions of **s**, of the different branches for $\mathbf{p = 1, 2, 5, 12}$, respectively

Comparison to field theory

So far no one has computed the partition function for any theory on a squashed Lens space

The only results for supersymmetric gauge theories on $\mathbf{S^3}/\mathbb{Z}_\mathbf{p}$ for $\mathbf{p} > \mathbf{1}$ are for the ABJM theory on the *round Lens space* [Alday-Fluder-Sparks]

The large $\mathbf{N}$ behaviour of the localized free energy agrees with the (naive) holographic free energy of $\mathrm{AdS}_\mathbf{4}/\mathbb{Z}_\mathbf{p}$ (green lines at $\mathbf{s} = \mathbf{1}/\mathbf{2}$ in the plots)

More generally, do our results predict the existence of new vacua in the field theories, and phase transitions among them?

Lifting bolts to M-theory

... although the Taub-Bolt-AdS solutions are smooth SUSY solutions of $\mathbf{d = 4}$ gauged supergravity, they do not always lift to SUSY solutions in $\mathbf{d = 11}$!

Consider the ABJM theory, which has internal space $\mathbf{Y_7 = S^7}$ (for $\mathbf{k = 1}$). Then the uplifting ansatz for the metric is

$$\mathbf{ds^2_{11} = R^2 \left[\tfrac{1}{4}ds^2_4 + (d\chi + \sigma + \tfrac{1}{2}A)^2 + ds^2_{\mathbb{CP}^3}\right]}$$

where $\mathbf{d\sigma =}$ Kähler form on $\mathbb{CP}^3$ and $\mathbf{A}$ is the $\mathbf{d = 4}$ (spin^c) gauge field

The coordinate χ has period $\mathbf{2\pi}$, but this then doesn't give a globally well-defined $\mathbf{d = 11}$ metric due to the $\pm\frac{1}{2}$ unit of flux of $\mathbf{F}$ through the bolt $\mathbf{S^2}$

Conclusions

- Gauge/gravity duality relates: supersymmetric gauge theories in curved backgrounds, localization, matrix models, quantum dylogarithm (and generalisations), geometric structures (e.g. Sasaki-Einstein manifolds)
- For the "biaxially squashed three spheres" we have found a complete agreement between large $\mathbf{N}$ limit of matrix models and supergravity solutions
- We obtained new non-trivial predictions for the large $\mathbf{N}$ limit of Lens space matrix models. In particular, we discovered a new type of supersymmetric "filling" with topology different from $\mathbb{R}^4$ with subtle properties
- The Taub-Bolt-AdS solutions only lift to well-behaved $\mathbf{d = 11}$ solutions only for certain choices of internal space $\mathbf{Y_7}$
- Can the Taub-Bolt-AdS solutions (and associated "phase transition" in $\mathbf{s}$) be understood in terms of a field theory computation on $\mathbf{S^3}/\mathbb{Z}_\mathbf{p}$?! Subtle global properties of gauge field in the boundary (Wilson lines) will play a role...